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# SPIN REPRESENTATIONS AND CENTRALIZER ALGEBRAS FOR $\text{Spin}(2n+1)$ (Topics in Young Diagrams and Representation Theory)

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# SPIN REPRESENTATIONS AND CENTRALIZER ALGEBRAS FOR $Spin(2n+1)$

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## 1. INTRODUCTION

These consecutive two articles are expositions of my manuscripts ([6], [7]). The references are in the last of the second exposition.)

Let  $G$  be Spin groups (Pin groups), namely the simply connected simple Lie groups of Type  $B_n$  or  $D_n$ , in other words, the double covering groups of  $SO(2n+1)$  or  $SO(2n)$  ( $O(2n+1)$  or  $O(2n)$  respectively). Then its character theory tells us that every irreducible spin representation of  $G$  ( a representation not coming from that of  $SO(2n+1)$

or  $SO(2n)$ ) can be realized in the tensor space  $\Delta \otimes^k V$  for some  $k$ , where  $\Delta$  is the fundamental spin representation of  $G$  and  $V = \mathbb{C}^N$  is the natural representation of  $O(N)$ . We consider the centralizer algebra  $\text{End}_G(\Delta \otimes^k V)$  of  $G$  on the tensor space  $\Delta \otimes^k V$  and give two kinds of explicit basis for  $\text{CS}_k$ . This algebra  $\text{CS}_k$  is a natural analogy of the Brauer centralizer algebra and contains the ordinary Brauer centralizer algebra from its definition. Finally we give an analogy of the Schur - Weyl duality in this case.

This is only the remaining case of realization of the irreducible representations of the classical groups in the tensor spaces, which are treated in the H. Weyl's book 「The Classical Groups」 ([9]). From now on, we state arguments over the field  $\mathbb{C}$ , or  $\mathbb{R}$ , but these work well over the field  $\mathbb{Q}$  or the field  $\mathbb{Q}(\sqrt{2})$  for Spin or Pin groups.

## 2. HISTORY AND MOTIVATION

Let us recall the classical situation.

Case 1.  $GL(n, \mathbb{C})$  (ref. 1901 I. Shur, 1939 H. Weyl, [9], Chap IV )

**Classical Schur - Weyl duality (or reciprocity)**

$$\begin{aligned} \text{End}_{GL(n)}(\otimes^k V) &= \langle \mathbb{C}[\mathfrak{S}_k] \rangle \\ \left( \begin{array}{c} \Longleftrightarrow \\ \text{Wedderburn's Theorem} \end{array} \right. & \text{End}_{\mathfrak{S}_k}(\otimes^k V) = \langle GL(n) \rangle \end{aligned}$$

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Here  $V = \mathbb{C}^n$  is the natural representation of  $GL(n)$  and  $\mathfrak{S}_k$  is the symmetric group of degree  $k$ , which acts on this space by the permutations of the positions of the tensor products. The bracket  $\langle \rangle$  denotes the enveloping algebra in  $\text{End}(\bigotimes^k V)$ .

Then we have

$$\bigotimes^k V = \sum_{\substack{\lambda: \text{partitions of size } k \\ \ell(\lambda) \leq n}} \lambda_{\mathfrak{S}_k} \otimes \lambda_{GL(n)}.$$

The projection from  $\bigotimes^k V$  to a specified irreducible representation  $\lambda_{GL(n)}$  is given by a Young symmetrizer defined by a standard Young Tableau of shape  $\lambda$ .

The underlying fact that all the irreducible polynomial representations occur in the space  $\bigotimes^k V$  comes from the following decomposition rule.

$$\lambda_{GL(n)} \otimes (1)_{GL(n)} = \sum_{\substack{\mu \supset \lambda, \ell(\mu) \leq n \\ |\mu/\lambda|=1}} \mu_{GL(n)}$$

In this case, we have a  $q$ -analog introduced by M. Jimbo. The quantum group of  $GL(n)$  and Iwahori-Hecke algebra of type  $A$  act on the space  $\bigotimes^k V$  as a dual pair.

Case 2.  $O(N, \mathbb{C})$  and  $Sp(2n, \mathbb{C})$  (ref. [9] Chap V, Chap VI)

We only state the case of  $O(N, \mathbb{C})$ . For  $Sp(2n, \mathbb{C})$ , the parallel argument goes well.

A natural analog of the argument of  $GL(n)$  is to consider the centralizer algebra  $\omega_k^N = \text{End}_{O(N)}(\bigotimes^k V)$ , where  $V = \mathbb{C}^N$  is the natural representation of  $G = O(N)$ . If we could know about the representation theory of  $\omega_k^N$  well, we would tell about the representation theory of  $G = O(N)$  just as in the Case 1 above. But this algebra is not so easy to be handled as H. Weyl called this algebra 'somewhat enigmatic algebra' in his book. So he took a short-cut to obtain the irreducible representations of  $O(N)$ .

Before to state it, we introduce the Brauer centralizer algebra. R. Brauer defined in 1937 ([2]) the source algebra of the centralizer algebras of  $O(N)$ , which is called now the Brauer centralizer algebra. We first define the Brauer diagrams. They are, by definition, the diagrams consisting of two lines of dots with  $k$  dots in each row, in which dots are connected with each other by edges and the edge multiplicity of each dot is exacty one. We denote the set of the above Brauer diagrams by  $\text{rk}$

Centralizer algebras for odd Spin

FIGURE 1. Brauer diagrams of  $k = 2$ 

Then the linear space  $Br_k(Q)$  over  $\mathbb{C}(Q)$  ( $Q$  :indeterminate) is defined by the formal sums of the formal basis elements consisting of all the Brauer diagrams  $B_k^k$ .

We make the above  $Br_k(Q)$  the algebra over  $\mathbb{C}(Q)$  by introducing the product rule as follows.

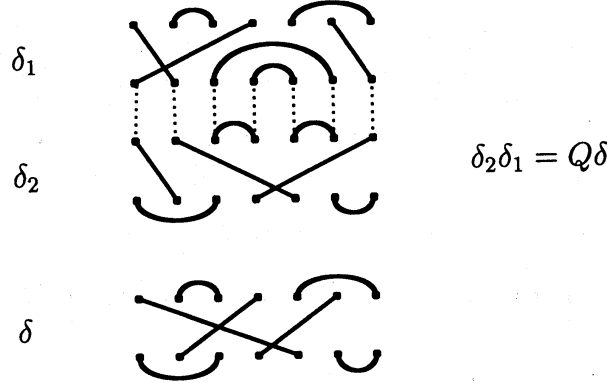


FIGURE 2. an example of the product rule of basis elements of a Brauer algebra

Generally as is given in the Figure 2, we connect two diagrams and in the conjunct diagram, let us denote the number of internal cycles by  $\gamma(\delta_2, \delta_1)$ . The result of the product is the scalar multiple by  $Q^{\gamma(\delta_2, \delta_1)}$  of 'the diagram obtained by connecting the top row with the bottom row in the conjunct diagram'.

The action of the Brauer algebra  $Br_k(Q)$  on  $\bigotimes^k V$  is given as follows. Let  $\dim V = N$  and we put  $Q = N$ . To illustrate the action, We write down the action of  $\delta_1$  in the Figure 2 on the space  $\bigotimes^7 V$  explicitly. Let  $G = O(7)$  and let  $(\ , \ )$  be the defining non-degenerate symmetric bilinear form of  $G$  and let  $\langle e_1, e_2, \dots, e_N \rangle$  be a base of  $V = \mathbb{C}^N$  and  $\langle e_1^*, e_2^*, \dots, e_N^* \rangle$  be its dual base.

$$\delta_1(v_1 \otimes v_2 \otimes \dots \otimes v_7) = (v_2, v_3)(v_5, v_7) \sum_{i,j} v_4 \otimes v_1 \otimes e_i \otimes e_j \otimes e_j^* \otimes e_i^* \otimes v_6$$

In the above, the operator which takes the inner product of the tensor components is called the contraction and for example, the operator corresponding to the inner product  $(v_2, v_3)$  is denoted by  $C_{2,3}$ . Also we denote the operator which embeds the invariant symmetric bilinear form  $\sum_i e_i \otimes e_i^*$  in the prescribed tensor positions  $k, \ell$  by  $\text{id}_V\{k, \ell\}$ . For example,  $\text{id}_V\{3, 6\}$  denotes the embedding  $\sum_i \cdot \otimes \cdot \otimes \underbrace{e_i}_{3} \otimes \cdot \otimes \cdot \otimes \underbrace{e_i^*}_{6} \otimes \cdot$ .

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Using these notations,  $\delta_1$  is given by

$$\delta_1 = \text{id}_V \{4,7\} \text{id}_V \{3,6\} \begin{pmatrix} 1 & 4 & 6 \\ 2 & 1 & 7 \end{pmatrix} C_{2,3} C_{5,7}.$$

Here  $\begin{pmatrix} 1 & 4 & 6 \\ 2 & 1 & 7 \end{pmatrix}$  denotes the partial permutation which sends the 1st component to the 2nd position and the 4th component to the 1st and the 6th component to the 7th.

We denote this representation of  $Br_k(N)$  by

$$\rho : Br_k(N) \rightarrow \text{End}(\bigotimes^k V)$$

Let us recall ‘The First Main Theorem’ and ‘The Second Main Theorem’ of the polynomial invariants for the orthogonal groups.

**Theorem 2.1** (H. Weyl The First Main Theorem 2.11A). *Let  $K$  be a field of characteristic 0 and let  $V = K^N$  be an  $N$ -dimensional vector space over  $K$ . By  $P(\bigoplus^k V)$ , we denote the polynomial ring over the linear space  $\bigoplus^k V$  and by  $\mathbf{v}_i$ , we denote the  $i$ th component of the direct summand.*

- (i) *The invariant polynomials  $P(\bigoplus^k V)^{O(N)}$  of  $O(N)$  is generated by the defining symmetric bilinear forms  $(\mathbf{v}_i, \mathbf{v}_j)$  ( $1 \leq i, j \leq k$ ) of  $O(N)$ .*
- (ii) *The relative invariant polynomials  $P(\bigoplus^k V)^{\text{rel}, O(N)}$  ( $= P(\bigoplus^k V)^{SO(N)}$ , i.e. the invariant polynomials of  $SO(N)$ ) of  $O(N)$  is generated by the defining symmetric bilinear forms  $(\mathbf{v}_i, \mathbf{v}_j)$  ( $1 \leq i, j \leq k$ ) and the determinants  $\det(\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \dots, \mathbf{v}_{i_N})$  ( $i_s \in [k]$ ).*

Since we have

$$\text{End}_{O(N)}(\bigotimes^k V) \cong (\bigotimes^k V \otimes \bigotimes^k V^*)^{O(N)} \cong (\bigotimes^{2k} V^*)^{O(N)}.$$

and can regard the right side of the above as the elements of  $P(\bigoplus^{2k} V)^{O(N)}$  which satisfy the multi-linear properties on each component. Then from the First Main Theorem, such elements can be written down as a sum of the elements of the form

$$(\mathbf{v}_{i_1}, \mathbf{v}_{i_2})(\mathbf{v}_{i_3}, \mathbf{v}_{i_4}) \cdots (\mathbf{v}_{i_{2k-1}}, \mathbf{v}_{i_{2k}}).$$

Here  $\{i_1, i_2, \dots, i_{2k}\} = \{1, 2, \dots, 2k\}$ .

The action of this element on the space  $\bigotimes^k V$  is :

$(\mathbf{v}_{i_1}, \mathbf{v}_{i_2})$  corresponds to  $\text{id}_V \{i_1, i_2\}$  for  $i_1, i_2 \leq k$  and  $C_{i_1-k, i_2-k}$  for  $i_1, i_2 > k$  and the partial permutation which sends the  $i_2 - k$ th component to the  $i_1$ th position for  $i_1 \leq k, i_2 > k$ .

From the First Main Theorem, the homom.  $\rho : Br_k(N) \rightarrow \text{End}(\bigotimes^k V)$  must be surjective.

Moreover if we recall the Second Main Theorem, we can show that  $\rho$  is injective in the case of  $N \geq k$ .

**Theorem 2.2** (H. Weyl The Second Main Theorem 2.17A ). *Let  $V$  and  $P(\bigoplus^k V)$  be as in the First Main Theorem.*

- (i) *The relations of the invariant polynomials  $(\mathbf{v}_i, \mathbf{v}_j)$  of  $O(N)$  in the algebra  $P(\bigoplus^k V)^{O(N)}$  are generated by the following determinants.*

$$(2.2.1) \quad \det \begin{pmatrix} (\mathbf{v}_{i_0}, \mathbf{v}_{j_0}) & (\mathbf{v}_{i_0}, \mathbf{v}_{j_1}) & \cdots & (\mathbf{v}_{i_0}, \mathbf{v}_{j_N}) \\ (\mathbf{v}_{i_1}, \mathbf{v}_{j_0}) & (\mathbf{v}_{i_1}, \mathbf{v}_{j_1}) & \cdots & (\mathbf{v}_{i_1}, \mathbf{v}_{j_N}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mathbf{v}_{i_N}, \mathbf{v}_{j_0}) & (\mathbf{v}_{i_N}, \mathbf{v}_{j_1}) & \cdots & (\mathbf{v}_{i_N}, \mathbf{v}_{j_N}) \end{pmatrix}$$

- (ii) *The relations of the invariant polynomials  $(\mathbf{v}_i, \mathbf{v}_j)$  and  $\det(\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \dots, \mathbf{v}_{i_N})$  of  $SO(N)$  are generated by the above relations (2.2.1) and the following relations:*

$$(2.2.2) \quad \det(\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \dots, \mathbf{v}_{i_N}) \det(S) \det(\mathbf{v}_{j_1}, \mathbf{v}_{j_2}, \dots, \mathbf{v}_{j_N}) - \det \begin{pmatrix} (\mathbf{v}_{i_1}, \mathbf{v}_{j_1}) & (\mathbf{v}_{i_1}, \mathbf{v}_{j_2}) & \cdots & (\mathbf{v}_{i_1}, \mathbf{v}_{j_N}) \\ (\mathbf{v}_{i_2}, \mathbf{v}_{j_1}) & (\mathbf{v}_{i_2}, \mathbf{v}_{j_2}) & \cdots & (\mathbf{v}_{i_2}, \mathbf{v}_{j_N}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mathbf{v}_{i_N}, \mathbf{v}_{j_1}) & (\mathbf{v}_{i_N}, \mathbf{v}_{j_2}) & \cdots & (\mathbf{v}_{i_N}, \mathbf{v}_{j_N}) \end{pmatrix} = 0$$

and

$$(2.2.3) \quad \sum_{k=0}^N (-1)^k \det(\mathbf{v}_{i_0}, \mathbf{v}_{i_1}, \dots, \widehat{\mathbf{v}_{i_k}}, \dots, \mathbf{v}_{i_N}) (\mathbf{v}_{i_k}, \mathbf{v}_j) = 0$$

Here  $S$  denotes the symmetric bilinear form corresponding to the inner product  $(\ , \ )$  and  $i_s, j \in [k]$ .

From the above, the minimum degree of the relations of the invariant polynomials of  $O(N)$  is  $2N + 2$ , so if  $2k < 2N + 2$ , i.e.,  $k \leq N$ ,  $\rho$  is injective.

**Remark 2.3.** *For the group  $Sp(2n)$ , in the definition of the products of the base elements of  $Br_k(2n)$ , we consider the contractions and the immersions by the defining alternating form of  $Sp(2n)$ , so we must add the signature to the product rules in the case of  $O(N)$  and the rest goes well.*

Then  $O(N)$  and  $\rho(Br_k(N))$  act on the space  $\bigotimes^k V$  as a dual pair, namely we have

$$\text{End}_{O(N)}(\bigotimes^k V) = \langle \rho(Br_k(N)) \rangle, \quad \text{End}_{\rho(Br_k(N))}(\bigotimes^k V) = \langle \mathbb{C}[O(N)] \rangle.$$

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However since the representation theory of  $\rho(Br_k(N)) = \omega_k^N$  is not easy to be understood, Weyl took the following way. Let  $T_k^0(V)$  be the intersection of the kernels of all the contractions  $\{C_{i,j}\} (1 \leq i < j \leq k)$  in the full tensor space  $T_k(V) = \bigotimes^k V$  of degree  $k$ .

Then on the space  $T_k^0(V)$ ,  $O(N)$  and the symmetric group  $\mathfrak{S}_k$  of degree  $k$  act as a dual pair. we note that  $Br_k(N)$  contains the group  $\mathfrak{S}_k$  naturally as the transpositions of the tensor components.

So we have the decomposition

$$T_k^0(V) = \sum_{\substack{\lambda: \text{partitions of size } k \\ \lambda'_1 + \lambda'_2 \leq N}} \lambda_{\mathfrak{S}_k} \otimes \lambda_{O(N)}.$$

, where  $\lambda'_1$  and  $\lambda'_2$  denote the lengths of the first and the second column of  $\lambda$  respectively. The Young diagrams satisfying the condition  $\lambda'_1 + \lambda'_2 \leq N$  are called 'permissible diagram'.

If  $\ell(\lambda) = \lambda'_1 \leq N/2$ ,  $\lambda_{O(N)}$  denotes the irreducible representation of  $O(N)$  with the height weight  $\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 + \dots + \lambda_n \epsilon_n$ . If  $\lambda$  is a permissible diagram and  $\lambda'_1 > N/2$ , let us put the Young diagram  $\bar{\lambda} = (N - \lambda'_1, \lambda'_2, \dots, \lambda'_n)'$  and call the irreducible representation  $\bar{\lambda}_{O(N)}$  the associate of  $\lambda_{O(N)}$ . Then we have  $\lambda_{O(N)} = \bar{\lambda}_{O(N)} \otimes \det$ .

The projection to an irred representation  $\lambda_{O(N)}$  in the space  $T_k^0(V)$  is given by a Young symmetrizer. The decomposition of the tensor product of  $\lambda_{O(N)}$  and the natural representation  $V = \mathbb{C}^N$  is given by

$$\lambda_O \otimes (1)_O = \sum_{\substack{\mu \supset \lambda \\ |\mu/\lambda|=1, \ell(\mu) \leq n}} \mu_O + \sum_{\substack{\lambda \supset \mu \\ |\delta/\mu|=1}} \mu_O.$$

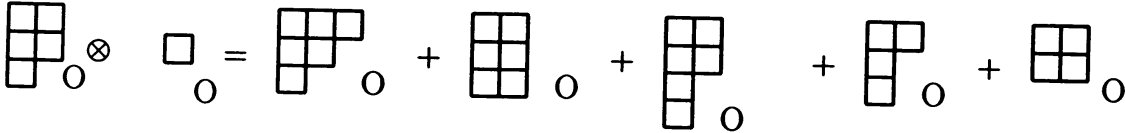


FIGURE 3. An example of the decomposition of the tensor product of  $O(N)$

This is a universal formula for  $O(N)$  and if  $N = 2n+1$  and  $\ell(\lambda) < n$ , we have

$$\lambda_{SO(2n+1)} \otimes (1)_{SO(2n+1)} = \sum_{\substack{\mu \supset \lambda \\ |\mu/\lambda|=1, \ell(\mu) \leq n}} \mu_{SO(2n+1)} + \sum_{\substack{\lambda \supset \mu \\ |\delta/\mu|=1}} \mu_{SO(2n+1)}.$$

If  $\ell(\lambda) = n$ , we have

$$\lambda_{SO(2n+1)} \otimes (1)_{SO(2n+1)} = \lambda_{SO(2n+1)} + \sum_{\substack{\mu \supset \lambda \\ |\mu/\lambda|=1, \ell(\mu) \leq n}} \mu_{SO(2n+1)} + \sum_{\substack{\lambda \supset \mu \\ |\delta/\mu|=1}} \mu_{SO(2n+1)}.$$

These formulas are the underlying fact that all the irreducible polynomial representations occur in the space  $\bigotimes^k V$ .

We have  $q$ - analog in this case too and the quantum group of type  $B_n$  and the  $q$ - analog of the Brauer centralizer algebra (Birman -Wenzl (-Murakami) algebra) act on the space  $\otimes^k V$  as a dual pair.

### 3. A SUMMARY OF REPRESENTATION OF $Spin(2n+1)$

We generalize the above constructions to the case of  $Spin(2n+1)$ , ( $Spin(2n)$ ,  $Pin(2n)$ ).

We state the theorems for  $G = Spin(2n+1)$ .

Let  $\Delta$  be the fundamental irreducible spin representation of  $Spin(2n+1)$  with the highest weight  $(1/2, 1/2, \dots, 1/2)$ . and for a partition  $\delta$  ( $\ell(\delta) \leq n$ ), let  $[\Delta, \delta]_{Spin(2n+1)}$  be the irreducible representation with the highest weight  $(1/2 + \delta_1, 1/2 + \delta_2, \dots, 1/2 + \delta_n)$ . We call these  $[\Delta, \delta]_{Spin(2n+1)}$ 's the irreducible spin representations of  $Spin(2n+1)$ .

We summarize the facts on the irreducible representations of  $Spin(2n+1)$ , which follow from its character theory.

**Theorem 3.1.** (i)

$$(3.1.1) \quad \Delta^2 = e_0 + e_1 + e_2 + \dots + e_n.$$

Here  $e_i$  denotes the exterior representation  $\bigwedge^i V$  of degree  $i$  of the natural representation  $V = \mathbb{C}^{2n+1}$ . Namely  $e_i = (1^i)_{SO(2n+1)}$ , ( $i = 1, 2, \dots, n$ ) and we have  $\bigwedge^i V \cong \bigwedge^{2n+1-i} V$ .

(ii)

$$(3.1.2)$$

$$[\Delta, \delta]_{Spin(2n+1)}(1)_{SO(2n+1)} = [\Delta, \delta]_{Spin(2n+1)} + \sum_{\substack{\mu \supset \delta \\ |\mu/\delta|=1, \ell(\mu) \leq n}} [\Delta, \mu]_{Spin(2n+1)} + \sum_{\substack{\delta \supset \mu \\ |\delta/\mu|=1}} [\Delta, \mu]_{Spin(2n+1)}.$$

(iii) For a partition  $\lambda$  ( $\ell(\lambda) \leq n$ ), we have

$$(3.1.3) \quad \Delta \otimes \lambda_{SO(2n+1)} = \sum_{\substack{\lambda \supset \mu \\ \lambda/\mu: \text{vertical strip}}} [\Delta, \mu]_{Spin(2n+1)}.$$

Therefore the irreducible representation  $\Delta$  occurs in the space  $\Delta \otimes \lambda_{SO(2n+1)}$  if and only if  $\lambda = (1^k)$ , ( $1 \leq k \leq n$ ). At that time the multiplicity is one and the exact decomposition is given as follows.

$$\Delta \otimes (1^k)_{SO(2n+1)} = \sum_{i=0}^k [\Delta, (1^i)]_{Spin(2n+1)}$$

From the above, we can conclude that every irreducible spin representation occurs in the space  $\Delta \otimes \bigotimes_k V$ . So we define the centralizer algebra  $CS_k$  by

$$CS_k = \text{End}_{Spin(2n+1)}(\Delta \otimes \bigotimes_k V)$$



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and call this algebra the spin centralizer algebra.

More generally we define the linear space  $\mathbf{CS}_1^k$  by

$$\mathbf{CS}_1^k = \text{Hom}_{\text{Spin}(2n+1)}(\Delta \otimes \otimes^k V, \Delta \otimes \otimes^l V).$$

Let us introduce the generalized Brauer diagram. The generalized Brauer diagrams are by definition, the diagrams of two lines dots with  $k$  dots in the upper row and  $l$  dots in the lower row, in which dots are connected with each other as in the usual Brauer diagrams except for admitting isolated points. Namely they are graphs with no loops in which the number of edges connected to each dot is either 0 or 1. We denote the set of the above diagrams by  $\mathbf{GB}_1^k$ .

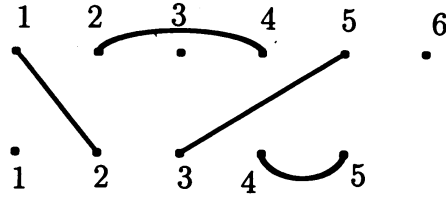


FIGURE 4. an example of the generalized Brauer diagrams with  $k = 6$  and  $l = 5$

When  $n \geq k$ , we will introduce two kind of basis, both of which are parametrized by  $\mathbf{GB}_k^k = \mathbf{GB}_k$  and one of which is the base coming from the invariant elements and the other of which is the base coming from the representation-theoretic manipulation. To distinguish them, we denote the base coming from the invariant elements by attaching the suffix 'inv' to the base element  $\mathbf{GB}_k$  and the base coming from the representation theory by attaching the suffix 'rt' to the base element  $\mathbf{GB}_k$ .

We will give the transformation rules between the above two basis and the decomposition rules of products of the base elements.

If we put  $2n + 1 = N = Q$  ( $Q$ : indeterminate) in the decomposition formulas of the products of the base elements, we can define the generic algebra  $\mathbf{CS}_k(Q)$  of the centralizer algebra  $\mathbf{CS}_k$  and  $\mathbf{CS}_k(Q) \supset Br_k(Q)$  holds naturally.

From the definition, we have

$$\mathbf{CS}_k = \text{End}_{\text{Spin}(2n+1)}(\Delta \otimes \otimes^k V) = (\Delta^* \otimes \otimes^k V^* \otimes \Delta \otimes \otimes^k V)^{\text{Spin}(2n+1)}.$$

To study the structure of the algebra  $\mathbf{CS}_k$ , we must give the following isomorphism explicitly.

$$\Delta^* \otimes \Delta \cong \Delta \otimes \Delta \cong \bigoplus_{i=0}^n \bigwedge^{2i} V \cong \bigoplus_{i=0}^n \bigwedge^{2i+1} V$$

(We note that for the group  $\text{Spin}(2n + 1)$ , we have  $\Delta^* \cong \Delta$ .)

First we give the actions of the Lie algebra  $Lie(Spin(2n+1)) = \mathfrak{so}(2n+1, S)$  on the base elements of the spaces  $\Delta$  and  $\bigwedge^i V$  explicitly. Here  $S$  is the defining nondegenerate symmetric bilinear form of  $O(2n+1)$ . We take a basis  $\langle u_1, u_2, \dots, u_n, u_0, u_{\bar{n}}, \dots, u_{\bar{1}} \rangle$  of  $V$  such that the matrix expression of  $S$  on this base is the anti-diagonal matrix  $S = (\delta_{i, 2n+2-i})$  and fix them hereafter. We introduce an order  $\{1 < 2 < \dots < n < 0 < \bar{n} < \dots < \bar{1}\}$  in the index set of the base elements.

From the definition we have

$$\mathfrak{so}(2n+1, S) = \{X \in M(2n+1, \mathbb{C}); XS + S^tX = 0\}$$

and we take a set of the simple root vectors as follows.

$$\begin{aligned} \text{ad}(X_k) &= E_{k, k+1} - E_{\bar{k}+1, \bar{k}}, & \text{ad}(X_n) &= \sqrt{2}(E_{n, 0} - E_{0, \bar{n}}), \\ \text{ad}(Y_k) &= E_{k+1, k} - E_{\bar{k}, \bar{k}+1}, & \text{ad}(Y_n) &= \sqrt{2}(E_{0, n} - E_{\bar{n}, 0}), \\ \text{ad}(h_i) &= E_{i, i} - E_{\bar{i}, \bar{i}}. \end{aligned}$$

Here  $k \in \{1, 2, \dots, n-1\}$  and  $i \in \{1, 2, \dots, n\}$ . For  $1 \leq k \leq n-1$ , we have  $[X_k, Y_k] = h_k - h_{k+1} = H_{\alpha_k}$ , ( $\alpha_k = \epsilon_k - \epsilon_{k+1}$ ) and  $[X_n, Y_n] = 2h_n = 2H_{\alpha_n}$ .

We take a basis of  $\Delta$  parametrized by all the subsets of  $[n] = \{1, 2, \dots, n\}$  and denote the basis elements by  $\{[I]\}$ , where  $I = \{i_1, i_2, \dots, i_r\}$  ( $1 \leq i_1 < i_2 < \dots < i_r \leq n$ ).

Namely we have  $\Delta = \bigoplus_{I \subset [n]} \mathbb{C}[I]$  and the action of Lie algebra  $\mathfrak{so}(2n+1, S)$  on this base is given as follows:

**Lemma 3.2.**

$$X_k[i_1, i_2, \dots, i_r] = \begin{cases} -[i_1, \dots, i_{s-1}, k+1, i_{s+1}, \dots, i_r] & \text{if } k = i_s \text{ and } k+1 < i_{s+1} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$X_n[i_1, i_2, i_3, \dots, i_r] = \begin{cases} -[i_1, i_2, i_3, \dots, i_{r-1}] & \text{if } i_r = n \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Y_k[i_1, i_2, \dots, i_r] = \begin{cases} -[i_1, \dots, i_{s-1}, k, i_{s+1}, \dots, i_r] & \text{if } k+1 = i_s \text{ and } k > i_{s-1} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Y_n[i_1, i_2, i_3, \dots, i_r] = \begin{cases} -[i_1, i_2, i_3, \dots, i_r, n] & \text{if } i_r \neq n \\ 0 & \text{otherwise,} \end{cases}$$

, where the sequence  $i_1, i_2, i_3, \dots, i_r$  are in the increasing order.

$$h_k[i_1, i_2, i_3, \dots, i_r] = \begin{cases} -\frac{1}{2}[i_1, i_2, i_3, \dots, i_r] & \text{if } k \in \{i_1, i_2, \dots, i_r\} \\ \frac{1}{2}[i_1, i_2, i_3, \dots, i_r] & \text{otherwise.} \end{cases}$$

Therefore  $[\emptyset]$  is the highest weight vector of  $\Delta$ . For convenience sake, we introduce the following convention. For any sequence  $i_1, i_2, i_3, \dots, i_r$  of positive integers, we define the corresponding element  $[i_1, i_2, i_3, \dots, i_r]$  in  $\Delta$  as follows.

$$[i_1, i_2, i_3, \dots, i_r] = \begin{cases} 0 & \text{if they are not distinct} \\ \epsilon(\sigma)[i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}, \dots, i_{\sigma(r)}], & \text{if they are distinct} \end{cases}$$

Here  $\sigma$  is the permutation of  $\{1, 2, \dots, r\}$  defined by the condition  $i_{\sigma(1)} < i_{\sigma(2)} < i_{\sigma(3)} < \dots < i_{\sigma(r)}$  and  $\epsilon(\sigma)$  denotes the signature of the permutation  $\sigma$ .

The compact real form  $\mathfrak{so}(2n+1)_{cpt}$  of  $\mathfrak{so}(2n+1, S)$  are generated over  $\mathbb{R}$  by the elements  $\sqrt{-1}h_i$ ,  $(i = 1, 2, \dots, n)$  and  $\sqrt{-1}(X_i + Y_i)$ ,  $X_i - Y_i$ ,  $(i = 1, 2, \dots, n)$ .

Then the invariant hermitian metrics of  $V$  and  $\Delta$  under the action of  $\mathfrak{so}(2n+1)_{cpt}$  are given such that the base  $\langle u_1, u_2, \dots, u_n, u_0, u_{\bar{n}}, \dots, u_{\bar{1}} \rangle$  of  $V$  and the base  $[I]_{I \subseteq [n]}$  of  $\Delta$  become orthonormal basis respectively.

#### 4. AN $\mathfrak{so}(2n+1)$ -EQUIVARIANT EMBEDDINGS FROM $\bigwedge^k V$ TO $\Delta^* \otimes \Delta$

We denote the natural base of the exterior product  $\bigwedge^r V$  by  $\langle i_1, i_2, \dots, i_r \rangle = u_{i_1} \wedge u_{i_2} \wedge \dots \wedge u_{i_r}$ ,  $i_k \in \{1, 2, \dots, n, 0, \bar{n}, \dots, \bar{1}\}$ .

Here we have

$$u_{i_1} \wedge u_{i_2} \wedge \dots \wedge u_{i_r} = \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} \epsilon(\sigma) u_{i_{\sigma^{-1}(1)}} \otimes u_{i_{\sigma^{-1}(2)}} \otimes \dots \otimes u_{i_{\sigma^{-1}(r)}}.$$

For  $I \subseteq [n] = \{1, 2, \dots, n\}$  with  $I = \{i_1, i_2, \dots, i_r\}$  ( $i_1 < i_2 < \dots < i_r$ ), we define the sequences by  $\underline{I} = \{i_1, i_2, \dots, i_r\}$  and  $\overleftarrow{I} = \{i_r, i_{r-1}, \dots, i_1\}$ .

Similarly we define the sequences by  $\overline{I} = \{\bar{i}_1, \bar{i}_2, \dots, \bar{i}_r\}$  and  $\overleftarrow{\overline{I}} = \{\bar{i}_r, \bar{i}_{r-1}, \dots, \bar{i}_1\}$ .

For any mutually disjoint sets  $I, J, W \subseteq [n]$ , we define the basis of the exterior algebras by

$\{\langle \underline{I}, \underline{W}, \overleftarrow{\underline{W}}, \overleftarrow{\underline{J}} \rangle, \langle \underline{I}, \underline{W}, 0, \overleftarrow{\underline{W}}, \overleftarrow{\underline{J}} \rangle\}$ . Here in the bracket, the juxtapositions of the index sets are considered a sequence as a whole.

We use the same convention for the basis  $\{[i_1, i_2, \dots, i_r]\}$  of  $\Delta$ . Namely we admit any sequence of positive integers in the bracket. Then  $\{[\underline{I}, \underline{K}] \otimes [\underline{J}, \underline{K}]^*\}$  becomes a basis of  $\Delta \otimes \Delta^*$ , where  $I, K, J$  run over all the mutually disjoint subsets of  $[n]$ .

We give an explicit embedding theorem of  $\bigwedge^k V$  in the space  $\Delta^* \otimes \Delta$ .

**Theorem 4.1.** *For  $k$  ( $1 \leq k \leq 2n+1$ ), there exists an  $\mathfrak{so}(2n+1)$ -equivariant embedding  $\phi_k$  of the space  $\bigwedge^k V$  into  $\Delta^* \otimes \Delta$  given as follows*

$$\phi_k(< \underline{J}, \underline{W}, \overline{\underline{W}}, \overline{\underline{I}} >) = \sum_{[n]-J-I \supseteq K} \frac{(-1)^{|\underline{W}-\underline{W} \cap \underline{K}|}}{2^{(n-|J|-|I|)/2}} [\underline{I}, \underline{K}] \otimes [\underline{J}, \underline{K}]^*$$

$$\phi_k(< \underline{J}, \underline{W}, 0, \overline{\underline{W}}, \overline{\underline{I}} >) = \sum_{[n]-J-I \supseteq K} \frac{(-1)^{|\underline{K}-\underline{K} \cap \underline{W}|}}{2^{(n-|J|-|I|)/2}} [\underline{I}, \underline{K}] \otimes [\underline{J}, \underline{K}]^*$$

Moreover the above  $\phi_k$  becomes an isometric embedding with respect to the invariant metrics.

From the above, the isomorphism  $\Delta \otimes \Delta^* \cong \bigoplus_{i=0}^n \bigwedge^{2i} V$  is given by

$$\phi_0 \oplus \phi_2 \oplus \dots \oplus \phi_{2n} : \bigoplus_{i=0}^n \bigwedge^{2i} V \longrightarrow \Delta \otimes \Delta^*.$$

We compare the same weight spaces in the both sides. For simplicity, we omit the  $\phi_k$ . Let  $I = \{i_1, i_2, \dots, i_r\}$  and  $J = \{j_1, j_2, \dots, j_s\}$  be mutually disjoint subsets of  $[n]$ .

By  $\epsilon_J - \epsilon_I$ , we denote the weight  $\epsilon_J - \epsilon_I = \epsilon_{j_1} + \epsilon_{j_2} + \dots + \epsilon_{j_s} - \epsilon_{i_1} - \epsilon_{i_2} - \dots - \epsilon_{i_r}$ . Then the base of the weight space with the weight  $\epsilon_J - \epsilon_I$  in the space  $\bigoplus_{i=0}^n \bigwedge^{2i} V$  is given by  $\{< \underline{J}, \underline{W}, \overline{\underline{W}}, \overline{\underline{I}} >\}$  if  $|J| + |I| \equiv 0 \pmod{2}$  and given by  $\{< \underline{J}, \underline{W}, 0, \overline{\underline{W}}, \overline{\underline{I}} >\}$  if  $|J| + |I| \equiv 1 \pmod{2}$  respectively. Here  $W$  runs over all the subsets of  $[n] - J - I$ .

Also the base of the weight space with the weight  $\epsilon_J - \epsilon_I$  in the space  $\Delta \otimes \Delta^*$  is given by  $\{[\underline{I}, \underline{K}] \otimes [\underline{J}, \underline{K}]^*\}$ , where  $K$  runs over all the subsets of  $[n] - J - I$ .

Since the above two basis are the parts of the orthonormal basis of the spaces  $\bigoplus_{i=0}^n \bigwedge^{2i} V$  and  $\Delta \otimes \Delta^*$  respectively, the transformation matrix

$\frac{1}{2^{(n-|J|-|I|)/2}} ((-1)^{|\underline{W}-\underline{W} \cap \underline{K}|})_{\underline{W}, \underline{K}}$  between them is a unitary matrix and its components are all real, so it becomes an orthogonal matrix. Therefore the matrix

$H_{J,I,n} = ((-1)^{|\underline{W}-\underline{W} \cap \underline{K}|})_{\underline{W}, \underline{K}}$  ( $\underline{W}, \underline{K} \subset [n] - J - I$ ) becomes an Hadamard matrix of size  $2^{[n]-J-I}$ . (The Hadamard matrix is, by definition, a matrix satisfying the conditions that all its components consist of  $\pm 1$  and that each row is orthogonal to all the other rows. For example an Hadamard matrix of size 2 is given by  $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ .) The inverse matrix of this matrix is given by its transposed matrix.

Therefore if  $|J| + |I| \equiv 0 \pmod{2}$ , we have

$$[\underline{I}, \underline{K}] \otimes [\underline{J}, \underline{K}]^* = \sum_{[n]-J-I \supseteq \underline{W}} \frac{(-1)^{|\underline{W}-\underline{W} \cap \underline{K}|}}{2^{(n-|J|-|I|)/2}} < \underline{J}, \underline{W}, \overline{\underline{W}}, \overline{\underline{I}} >$$

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and if  $|J| + |I| \equiv 1 \pmod{2}$ , we have

$$[\underline{J}, \underline{K}] \otimes [\underline{J}, \underline{K}]^* = \sum_{[n]-J-I \supseteq W} \frac{(-1)^{|K-K \cap W|}}{2^{(n-|J|-|I|)/2}} < \underline{J}, \underline{W}, 0, \overline{\underline{W}}, \overline{\underline{I}} >.$$

## 5. AN INVARIANT THEORETIC PARAMETERIZATION

We will be back to the Invariant theory. Since

$$\text{End}_{\text{Spin}(2n+1)}(\Delta \otimes \otimes^k V) = (\Delta^* \otimes \Delta \otimes \otimes^{2k} V)^{\text{Spin}(2n+1)} = (\oplus_{i=0}^n \wedge^{2i} V \otimes \otimes^{2k} V)^{\text{Spin}(2n+1)},$$

it is enough to obtain an explicit base of the invariant polynomials in the space

$$(\wedge^r V^* \otimes \otimes^s V^*)^{SO(2n+1)} \subset (\otimes^{r+s} V^*)^{SO(2n+1)} \subset P(\oplus^{r+s} V)^{SO(2n+1)}.$$

We can assume  $r + s \equiv 0 \pmod{2}$ . Those basis elements are multilinear in each variables and has the alternating properties in the first  $r$  variables, regarded as the elements of  $P(\oplus^{r+s} V)^{SO(2n+1)}$ .

The degree of the determinant polynomial is  $2n + 1$  and its parity is odd, so its multiplicity must be even as the element of the  $(\otimes^{r+s} V^*)^{SO(2n+1)} \subset P(\oplus^{r+s} V)$ , since  $r + s \equiv 0 \pmod{2}$ . The formula (2.2.2) of the Second Main Theorem 2.2 tells us that the invariant polynomials are generated by  $(\mathbf{u}, \mathbf{v})$ .

First we write down the elements of  $(\wedge^r V^* \otimes \otimes^s V^*)^{SO(2n+1)}$  ( $r + s \equiv 0 \pmod{2}$ ). If  $s < r$ ,  $\otimes^s V$  can not contain  $\wedge^r V$ , so this space must be 0. Let us assume that  $s \geq r$ .

Let  $\mathbf{t} = \{t_1, t_2, \dots, t_r\}$  ( $t_1 < t_2 < \dots < t_r$ ) and  $\mathbf{m} = \{m_1, \dots, m_u\}$  and  $\mathbf{l} = \{l_1, \dots, l_u\}$  be ordered index sets (or sequences) such that as sets, they are mutually disjoint and satisfy the condition  $[s] = \mathbf{t} \sqcup \mathbf{m} \sqcup \mathbf{l}$ . By  $\{\mathbf{m}, \mathbf{l}\}$ , we denote the  $u = \frac{s-r}{2}$  pairs of indices  $\{\mathbf{m}, \mathbf{l}\} = \{\{m_1, l_1\}, \{m_2, l_2\}, \dots, \{m_u, l_u\}\}$ .

From the definition,  $\mathbf{t}$  must satisfy  $|\mathbf{t}| \leq 2n + 1$ . So the invariant polynomials can be written as sums of the following polynomials:

$$T_{\mathbf{t}, \{\mathbf{m}, \mathbf{l}\}} = \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} \epsilon(\sigma) (x_{\sigma^{-1}(1)}, y_{t_1}) (x_{\sigma^{-1}(2)}, y_{t_2}) \dots (x_{\sigma^{-1}(r)}, y_{t_r}) \times \prod_{j=1}^u (y_{m_j}, y_{l_j}).$$

Here  $x_j$  ( $j = 1, 2, \dots, r$ ) denotes the variables of the first  $r$  tensor components in the space  $\otimes^{r+s} V$  and  $y_j$  ( $j = 1, 2, \dots, s$ ) denotes the variables of the last  $s$  tensor components.

**Lemma 5.1.** *Let  $\text{CS}_1^k = \text{Hom}_{\text{Spin}(2n+1)}(\Delta \otimes \otimes^k V, \Delta \otimes \otimes^l V)$ . If  $s = k + l \equiv 0 \pmod{2}$ , we allow only the  $\mathbf{t}$ 's satisfying the conditions that*

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$|t| \leq 2n+1$  and  $|t| \equiv 0 \pmod{2}$ . If  $s = k+l \equiv 1 \pmod{2}$ , we allow only the  $t$ 's satisfying the conditions that  $|t| \leq 2n+1$  and  $|t| \equiv 1 \pmod{2}$ .

Then the above invariant polynomials  $T_{t,\{m,l\}}$  span linealy the space  $CS_1^k$ . Namely,

$$CS_1^k = \sum_{t \sqcup m \sqcup l = [s]} \mathbb{C} T_{t,\{m,l\}}.$$

Moreover if  $s = k+l < 2n+1$ , the elements  $T_{t,\{m,l\}}$  ( $t \sqcup m \sqcup l = [s]$ ) are linealy independent, i.e.,

$$CS_1^k = \bigoplus_{t \sqcup m \sqcup l = [s]} \mathbb{C} T_{t,\{m,l\}}.$$

There exists a natural correspondence between the generalized Brauer diagrams  $GB_1^k$  and the polynomials  $T_{t,\{m,l\}}$ . That is, the part  $\{m,l\}$  corresponds to the ordinary Brauer diagram whose edges are given by the pairs in  $\{m,l\}$  and the part  $\{t\}$  corresponds to the isolated points. We denotes these elements by adding the suffix 'inv' to the diagrams  $GB_1^k$ . (We can write down the action of this element on the tensor space explicitly.)

## 6. A REPRESENTATION THEORETIC PARAMETERIZATION

Let us recall the formula  $\Delta \otimes \bigwedge^k V = \sum_{i=0}^k [\Delta, (1^i)]_{Spin(2n+1)}$  in Theorem 3.1.

Hence we have  $\dim(\text{Hom}_{Spin(2n+1)}(\Delta \otimes \bigwedge^k V, \Delta)) = 1$  and  $\dim(\text{Hom}_{Spin(2n+1)}(\Delta, \Delta \otimes \bigwedge^k V)) = 1$ .

1. Then the  $\mathfrak{so}(2n+1)$ -equivariant projection  $\text{pr}_k : \Delta \otimes \bigwedge^k V \rightarrow \Delta$  and the  $\mathfrak{so}(2n+1)$ -equivariant injection  $\text{inj}_k : \Delta \rightarrow \Delta \otimes \bigwedge^k V$  can be given as follows (up to constant).

**Definition 6.1.**

$$\begin{aligned} & \text{pr}_k([\underline{T}_\rightarrow] \otimes \langle \underline{I}_\rightarrow, \underline{W}_\rightarrow, \overline{\underline{W}}, \overline{\underline{J}} \rangle) \\ &= \begin{cases} 0 & \text{if } I \not\subseteq T, \\ \epsilon \left( \begin{pmatrix} \underline{T}_\rightarrow & \\ \underline{I}_\rightarrow & \underline{K}_\rightarrow \end{pmatrix} \right) (-1)^{|\underline{W}-\underline{W} \cap \underline{K}|} 2^{(|I|+|J|)/2} [\underline{J}_\rightarrow \underline{K}_\rightarrow] & \text{if } I \subseteq T, \end{cases} \end{aligned}$$

and

$$\begin{aligned} & \text{pr}_k([\underline{T}_\rightarrow] \otimes \langle \underline{I}_\rightarrow, \underline{W}_\rightarrow, 0, \overline{\underline{W}}, \overline{\underline{J}} \rangle) \\ &= \begin{cases} 0 & \text{if } I \not\subseteq T, \\ \epsilon \left( \begin{pmatrix} \underline{T}_\rightarrow & \\ \underline{I}_\rightarrow & \underline{K}_\rightarrow \end{pmatrix} \right) (-1)^{|\underline{K}-\underline{W} \cap \underline{K}|} 2^{(|I|+|J|)/2} [\underline{J}_\rightarrow \underline{K}_\rightarrow] & \text{if } I \subseteq T. \end{cases} \end{aligned}$$

Here we put  $K = T - I$  and  $\epsilon \left( \begin{pmatrix} \underline{T}_\rightarrow & \\ \underline{I}_\rightarrow & \underline{K}_\rightarrow \end{pmatrix} \right)$  denotes the signature of the permutation which sends  $\underline{T}_\rightarrow$  to  $\underline{I}_\rightarrow \underline{K}_\rightarrow$ .

**Definition 6.2.**

$$\begin{aligned} \text{inj}_k([\mathcal{T}_\downarrow]) = & \sum_{\substack{\mathcal{I} \subseteq \mathcal{T} \\ k = |\mathcal{T}| - |\mathcal{I}|}} \epsilon \left( \begin{array}{c} \mathcal{T}_\downarrow \\ \mathcal{I}_\downarrow \end{array} \middle| \mathcal{K}_\downarrow \right) \left( \sum_{\substack{\mathcal{J} \subseteq ([n] - \mathcal{T}) \\ \mathcal{W} \subseteq ([n] - \mathcal{I} - \mathcal{J}) \\ |\mathcal{J}| + |\mathcal{I}| + 2|\mathcal{W}| = k}} (-1)^{|\mathcal{W} - \mathcal{W} \cap \mathcal{K}|} 2^{(|\mathcal{J}| + |\mathcal{I}|)/2} [\mathcal{J}_\downarrow, \mathcal{K}_\downarrow] \otimes k! < \mathcal{J}_\downarrow, \mathcal{W}_\downarrow, \overline{\mathcal{W}}, \right. \\ & + \sum_{\substack{\mathcal{J} \subseteq ([n] - \mathcal{T}) \\ \mathcal{W} \subseteq ([n] - \mathcal{I} - \mathcal{J}) \\ |\mathcal{J}| + |\mathcal{I}| + 2|\mathcal{W}| + 1 = k}} (-1)^{|\mathcal{K} - \mathcal{W} \cap \mathcal{K}|} 2^{(|\mathcal{J}| + |\mathcal{I}|)/2} [\mathcal{J}_\downarrow, \mathcal{K}_\downarrow] \otimes k! < \mathcal{J}_\downarrow, \mathcal{W}_\downarrow, 0, \overline{\mathcal{W}}, \overline{\mathcal{I}} > \left. \right). \end{aligned}$$

For an index set  $\mathcal{T} = \{t_1, t_2, \dots, t_p\}$  ( $1 \leq t_1 < t_2 < \dots < t_p \leq k$ ), we define the projection  $\text{pr}_{\mathcal{T}} : \Delta \otimes \otimes^k V \rightarrow \Delta \otimes \otimes^{k-p} V$  as follows.

We prepare a notation.

**Definition 6.3.** Let  $\text{Alt}_{\mathcal{T}}$  be the alternating operator on the tensor components which sit in the positions indexed by  $\mathcal{T}$  in the space  $\otimes^k V$ . That is,

$$\begin{aligned} \text{Alt}_{\mathcal{T}}(v_1 \otimes v_2 \otimes \dots \otimes v_k) = & \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \epsilon(\sigma) v_1 \otimes \dots \otimes v_{t_{\sigma^{-1}(1)}} \otimes \dots \otimes v_{t_{\sigma^{-1}(2)}} \otimes \dots \otimes v_{t_{\sigma^{-1}(p)}} \otimes \dots \otimes v_k. \end{aligned}$$

For any index set  $\mathcal{T}$ , we define the operator  $\text{Alt}_{\mathcal{T}}$  such that it has the alternating properties on the index set  $\mathcal{T}$ . Namely for any  $\sigma \in \mathfrak{S}_p$  and for any sequence of positive integers  $\mathcal{T} = \{t_1, t_2, \dots, t_p\}$ , we define  $\text{Alt}_{\sigma(\mathcal{T})} = \text{Alt}_{\{t_{\sigma^{-1}(1)}, t_{\sigma^{-1}(2)}, \dots, t_{\sigma^{-1}(p)}\}} = \epsilon(\sigma) \text{Alt}_{\mathcal{T}}$ .

**Definition 6.4.** Let  $\text{pr}_{\mathcal{T}} : \Delta \otimes \otimes^k V \rightarrow \Delta \otimes \otimes^{k-p} V$  be the projection map obtained by the composition of the map  $\text{Alt}_{\mathcal{T}}$  and  $\text{pr}_p$ , i.e.,  $\text{pr}_{\mathcal{T}} = \text{pr}_p \circ \text{Alt}_{\mathcal{T}}$ . Here  $\text{pr}_p$  acts on the alternating tensors sitting in the positions indexed by  $\mathcal{T}$  in the space  $\Delta \otimes \otimes^k V$ .

From the definition, we have  $\text{pr}_{\mathcal{T}} \in \text{Hom}_{\text{Spin}(2n+1)}(\Delta \otimes \otimes^k V, \Delta \otimes \otimes^{k-p} V)$  and it has the alternating property on the index set  $\mathcal{T}$ .

Similarly we define the  $\mathfrak{so}(2n+1)$ -equivariant embedding  $\text{inj}_{\mathcal{T}} \in \text{Hom}_{\text{Spin}(2n+1)}(\Delta \otimes \otimes^{k-p} V, \Delta \otimes \otimes^k V)$  as follows.

**Definition 6.5.** Let  $\text{inj}_{\mathcal{T}} : \Delta \otimes \otimes^{k-p} V \rightarrow \Delta \otimes \otimes^k V$  be the immersion obtained by the composition of the map  $\text{inj}_p : \Delta \rightarrow \Delta \otimes \bigwedge^p V$  and the linear embedding of the resulting tensors in the positions indexed by  $\mathcal{T}_\downarrow$ . Namely the embedding is the map which sends the first component of the alternating tensors  $< \mathcal{J}_\downarrow, \mathcal{W}_\downarrow, \overline{\mathcal{W}}, \overline{\mathcal{I}} >$  to the  $t_1$ th position in the space  $\Delta \otimes \otimes^{k-p} V$  and the second component to  $t_2$ th position in the

space  $\Delta \otimes \otimes^{k-p+1} V$  and so on. We denote this embedding of the alternating tensor  $\langle \underline{J}, \underline{W}, \underline{\overline{W}}, \underline{\overline{I}} \rangle$  by  $\langle \underline{J}, \underline{W}, \underline{\overline{W}}, \underline{\overline{I}} \rangle_{\mathbf{T}}$ .

From the definition,  $\text{inj}_{\mathbf{T}}$  has the alternating property on the index set  $\mathbf{T}$  too. We define the elements of  $\text{CS}_1^k$  parametrized by the generalized Brauer diagrams  $\text{GB}_1^k$ , coming from the representation theory as follows. We fix an element of the diagrams  $\text{GB}_1^k$ , and let  $\mathbf{T}_u$  be its isolated points in the upper row and  $\mathbf{T}_l$  be its isolated points in the lower row.

Then the action represented by the isolated points in the upper row corresponds to the projection  $\text{pr}_{\mathbf{T}_u}$  and the action represented by the isolated points in the lower row corresponds to the immersion  $\text{inj}_{\mathbf{T}_l}$ . Namely the total action represented by the isolated points corresponds to the composition map

$$\Delta \otimes \otimes^k V \xrightarrow{\text{pr}_{\mathbf{T}_u}} \Delta \otimes \otimes^{k-p} V \xrightarrow{\text{inj}_{\mathbf{T}_l}} \Delta \otimes \otimes^k V.$$

Finally we define the action corresponding to the points which are not isolated just in the same way as those of the ordinary Brauer diagrams.

We denote these elements by adding the suffix 'rt' to the diagrams of  $\text{GB}_1^k$ .

Whether these elements span linearly the space  $\text{CS}_1^k$  or not, or whether these elements become a base or not is not clear at present. We show in the next section that if  $k \leq n$  and  $l \leq n$ , we give the explicit relations between two parametrizations and that they become a base in this case.

## 7. RELATION BETWEEN TWO PARAMETERIZATION

Since the difference between two parametrizations are only in the actions corresponding to the isolated points, we give the relations between them. Let  $\mathbf{T}_u$  ( $|\mathbf{T}_u| = p$ ) be the isolated points in the upper row and let  $\mathbf{T}_l$  ( $|\mathbf{T}_l| = q$ ) be the isolated points in the lower row.

We denote the homomorphism  $\Delta \otimes \otimes^p V \rightarrow \Delta \otimes \otimes^q V$ , determined by the invariant polynomial by  $\psi_{\mathbf{T}_l}^{\mathbf{T}_u}$ , or simply by  $\psi_q^p$  if the isolated points are tacitly understood. Here the invariant polynomial which we consider in the above is given by

$$\sum_{\sigma \in \mathfrak{S}_r} \epsilon(\sigma) (x_{\sigma^{-1}(1)}, y_{t_1}) (x_{\sigma^{-1}(2)}, y_{t_2}) \cdots (x_{\sigma^{-1}(r)}, y_{t_r}).$$

For any  $\sigma \in \mathfrak{S}_k$  and  $\tau \in \mathfrak{S}_l$ , we have  $\tau \circ \psi_{\mathbf{T}_l}^{\mathbf{T}_u} \circ \sigma = \psi_{\tau(\mathbf{T}_l)}^{\sigma^{-1}(\mathbf{T}_u)}$ . So it is enough to give the explicit description for  $\psi_{[q]}^{[p]}$  in terms of the representation theoretical operators, where  $[p] = \{1, 2, \dots, p\}$  and  $[q] = \{1, 2, \dots, q\}$ .



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So we consider this element as the invariant polynomial in the space  $(\bigwedge^{p+q} V)^* \otimes \bigwedge^{p+q} V$ .

The relation of the above two actions are given as follows.

**Theorem 7.1.** *If  $p \leq n$  and  $q \leq n$ , then we have*

(7.1.1)

$$\psi_{[q]}^{[p]} = \sum_{i=0}^{\min(p,q)} (-1)^{(p-i)(q-i)} \sum_{\substack{\sigma \in \mathfrak{S}_q \\ \tau \in \mathfrak{S}_p}} \epsilon(\sigma) \epsilon(\tau) \frac{\text{inj}_{\{\sigma([i+1,q])\}}}{(q-i)!} \frac{\binom{\tau([1,i])}{\sigma([1,i])}}{i!} \frac{\text{pr}_{\{\tau([i+1,p])\}}}{(p-i)!}$$

and

(7.1.2)

$$\text{inj}_{[q]} \circ \text{pr}_{[p]} = \sum_{i=0}^{\min(p,q)} (-1)^{i+pq} \sum_{\substack{\sigma \in \mathfrak{S}_q \\ \tau \in \mathfrak{S}_p}} \epsilon(\sigma) \epsilon(\tau) \frac{\psi_{\{\sigma([i+1,q])\}}^{\{\tau([i+1,p])\}}}{(q-i)!(p-i)!} \otimes \frac{\binom{\tau([1,i])}{\sigma([1,i])}}{i!}.$$

Here  $\sigma([i+1, q]) = \{\sigma(i+1), \sigma(i+2), \dots, \sigma(q)\}$  and  $\tau([i+1, p]) = \{\tau(i+1), \tau(i+2), \dots, \tau(p)\}$  and  $\binom{\tau([1, i])}{\sigma([1, i])}$  denotes the partial permutation which sends the  $\tau(u)$ -component ( $u = 1, 2, \dots, i$ ) of the upper row  $\Delta \otimes \bigotimes^p V$  to the  $\sigma(u)$ th component of the lower row  $\Delta \otimes \bigotimes^q V$ .

Hence if  $k \leq n$  and  $l \leq n$ , the elements  $\{D_{rt}\}_{D \in \mathbf{GB}_1^k}$  coming from the representation theory also become a base of  $\mathbf{CS}_1^k$ .

If  $p > n$ , or if  $q > n$ , (we always assume that  $p + q \leq 2n + 1$ .) we have have similar theorems to the above.

**Remark 7.2.** *The righthand of the formula (7.1.1) can be considered as the composition of the homomorphisms, but the righthand of the second formula (7.1.2) is not the composition of homomorphisms and it expresses a homomorphism as a whole, so we put the tensor symbol  $\otimes$  in the middle.*

We give an example of the transformation between two parametrization.

## 8. RELATIONS BETWEEN $Spin(2n+1)$ -EQUIVARIANT HOMOMORPHISMS

In this section we give the explicit relations between the  $Spin(2n+1)$ -equivariant homomorphisms  $\text{pr}$ ,  $\text{inj}$ , contraction operators and the immersion of the invariant symmetric bilinear form  $S$ . Using these formulas we can deduce the product formulas of the generalized Brauer diagrams. By  $C_{\{i,j\}}$ , we denote the contraction by  $S$  of the  $i$ th and  $j$ th tensor components and by  $\text{id}_V_{\{i,j\}}$  denote the immersion of the invariant

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$$\begin{array}{c}
\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} = \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} - \begin{array}{c} | \\ | \\ | \end{array} \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} + \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} + \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
\text{inv} \qquad \qquad \text{rt} \qquad \qquad \text{rt} \qquad \qquad \text{rt} \qquad \qquad \text{rt} \\
\\
- \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \begin{array}{c} | \\ | \\ | \end{array} + \begin{array}{c} | \\ | \\ | \end{array} \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} - \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \\
\qquad \qquad \qquad \text{rt} \qquad \qquad \text{rt} \qquad \qquad \text{rt} \\
\\
\begin{array}{c} | \\ | \\ | \end{array} \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} = - \begin{array}{c} | \\ | \\ | \end{array} \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} + \begin{array}{c} | \\ | \\ | \end{array} \begin{array}{c} | \\ | \\ | \end{array} \\
\text{inv} \qquad \qquad \text{rt} \qquad \qquad \text{rt}
\end{array}$$

FIGURE 5. an example of the transformation between two parametrization when  $n \geq 2$

form  $S$  to the  $i$ th and  $j$ th components. Then we have the following formulas.

**Theorem 8.1.** (i) If  $p \leq n$ , as a homomorphism from  $\Delta \otimes \otimes^p V$  to  $\Delta$  ( where we consider the tensor  $\otimes^p V$  sits in the positions  $\{q+1, q+2, \dots, p+q\}$ . ), we have

$$(8.1.1) \quad \text{pr}_{\{[1, q+p]\}} \circ \text{inj}_{\{[1, q]\}} = (2n+1-p)_q \text{pr}_{\{[q+1, q+p]\}}.$$

Here  $(2n+1-p)_q$  denotes the lower factorial, namely for any  $x$  and any nonnegative integer  $i$ ,  $(x)_i = x(x-1)(x-2) \cdots (x-(i-1))$ . and we put  $[1, p] = \{1, 2, \dots, p\}$  ( as a sequence ).

Moreover if  $p = 0$ , we consider  $\text{pr} = \text{identity map of } \Delta$ .

(ii) If  $p \leq n$ , as a homomorphism from  $\Delta$  to  $\Delta \otimes \otimes^p V$  ( where we consider the tensor  $\otimes^p V$  sits in the positions  $\{q+1, q+2, \dots, p+q\}$ . ), we have

$$(8.1.2) \quad \text{pr}_{\{[1, q]\}} \circ \text{inj}_{\{[1, q+p]\}} = (2n+1-p)_q \text{inj}_{\{[q+1, q+p]\}}.$$

(iii) If  $p \leq n$  and  $q \leq n$ , as a homomorphism from  $\Delta$  to  $\Delta \otimes \otimes^{p+q} V$ , we have

$$(8.1.3)$$

$$\text{inj}_{\{[1, q]\}} \circ \text{inj}_{\{[q+1, q+p]\}} =$$

$$\sum_{i=0}^{\min(p, q)} (-1)^{qi + \binom{i+1}{2}} \sum_{\substack{\sigma \in \mathfrak{S}_q \\ \tau \in \mathfrak{S}_p[q]}} \epsilon(\sigma) \epsilon(\tau) \frac{\prod_{u=1}^i \text{id}_{V_{\{\sigma(u), \tau(q+u)\}}}}{i!} \frac{\text{inj}_{\{\sigma([i+1, q]), \tau([q+i+1, q+p])\}}}}{(q-i)!(p-i)!}.$$

Here  $\mathfrak{S}_p[q]$  denotes the symmetric group acting on the set  $\{q+1, q+2, \dots, q+p\}$  and  $\sigma([i+1, q]) = \{\sigma(i+1), \sigma(i+2), \dots, \sigma(q)\}$  and  $\tau([q+i+1, p]) = \{\tau(q+i+1), \tau(q+i+2), \dots, \tau(q+p)\}$ .

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(iv) If  $p \leq n$  and  $q \leq n$ , as a homomorphism from  $\Delta \otimes \otimes^{p+q} V$  to  $\Delta$ , we have

(8.1.4)

$$\text{pr}_{\{[1,q]\}} \circ \text{pr}_{\{[q+1,q+p]\}} = \sum_{i=0}^{\min(p,q)} (-1)^{qp+qi+\binom{i}{2}} \sum_{\substack{\sigma \in \mathfrak{S}_q \\ \tau \in \mathfrak{S}_p[q]}} \epsilon(\sigma) \epsilon(\tau) \frac{\text{pr}_{\{\sigma([i+1,q]), \tau([q+i+1,q+p])\}}}{(q-i)!(p-i)!} \frac{\prod_{u=1}^i C_{\{\sigma(u), \tau(q+u)\}}}{i!}.$$

(v) If  $p \geq t \geq 0$  and  $p-t \leq n$  and  $q \leq n$ , as a homomorphism from  $\Delta \otimes \otimes^{p-t} V$  to  $\Delta \otimes \otimes^q V$  ( where we consider the tensor  $\otimes^{p-t} V$  sits in the positions  $\{q+t+1, q+t+2, \dots, q+p\}$  ), we have

(8.1.5)

$$\text{pr}_{\{[q+1,q+p]\}} \circ \text{inj}_{\{[1,q+t]\}} = \sum_{i=0}^{\min(p-t,q)} (-1)^{(q-i)(p-i)+it} \times \left( \sum_{u=0}^i \binom{i}{u} (2n+1-p-q+t+i-u)_t \right) \sum_{\substack{\sigma \in \mathfrak{S}_q \\ \tau \in \mathfrak{S}_{p-t}[q+t]}} \epsilon(\sigma) \epsilon(\tau) \frac{\text{inj}_{\{\sigma([i+1,q])\}}}{(q-i)!} \frac{\binom{\tau([q+t+1, q+t+i])}{\sigma([1,i])}}{i!} \frac{\text{pr}_{\{\tau([q+t+i+1, q+p])\}}}{(p-t-i)!}.$$

Here the  $\binom{i}{u}$  in the paren denotes the ordinary binomial coefficient. If  $t=0$ , then  $(2n+1-p-q+0+i-u)_0 = 1$ , the sum in the paren is equal to  $2^i$ .

(vi) If  $p \leq n$  and  $q \leq n$ , as a homomorphism from  $\Delta \otimes \otimes^q V$  to  $\Delta \otimes \otimes^p V$  ( where we consider the tensor sits  $\otimes^q V$  in the positions  $\{p+q+1, p+q+2, \dots, p+2q\}$  ), we have

$$(8.1.6) \quad \prod_{i=1}^q C_{\{i, p+q+i\}} \text{inj}_{\{[1, q+p]\}} = \sum_{i=0}^{\min(p,q)} (-1)^{pq+\binom{q}{2}+i(p+q-1)} \sum_{\substack{\sigma \in \mathfrak{S}_q[q+p] \\ \tau \in \mathfrak{S}_p[q]}} \epsilon(\sigma) \epsilon(\tau) \frac{\text{inj}_{\{\tau([q+i+1, q+p])\}}}{(p-i)!} \times \frac{\binom{\sigma([p+q+1, p+q+i])}{\tau([q+1, q+i])}}{i!} \frac{\text{pr}_{\{\sigma([q+p+i+1, 2q+p])\}}}{(q-i)!}.$$

(vii) If  $p \leq n$  and  $q \leq n$ , as a homomorphism from  $\Delta \otimes \otimes^q V$  to  $\Delta \otimes \otimes^p V$  ( where we consider the tensor sits  $\otimes^q V$  in the positions  $[q]$  and the tensor  $\otimes^p V$  sits in the positions  $\{p+q+1, p+$

$q + 2, \dots, 2p + q\}$ ), we have

$$(8.1.7) \quad \text{pr}_{\{[1, q+p]\}} \prod_{i=1}^p \text{id}_{V_{\{q+i, p+q+i\}}} = \sum_{i=0}^{\min(p, q)} (-1)^{\binom{p}{2} + i(p+q+1)} \sum_{\substack{\sigma \in \mathfrak{S}_q \\ \tau \in \mathfrak{S}_{p[q+p]}}} \epsilon(\sigma) \epsilon(\tau) \frac{\text{inj}_{\{\tau([p+q+i+1, q+2p])\}}}{(p-i)!} \times \frac{\binom{\sigma([1, i])}{\tau([p+q+1, p+q+i])}}{i!} \frac{\text{pr}_{\{\sigma(i+1), \sigma(i+2), \dots, \sigma(q)\}}}{(q-i)!}.$$

**Remark 8.2.** If we exchange  $2n + 1$  for an indeterminate  $X$  simultaneously in the above formulas, we can define the ‘generic’ centralizer algebra of  $\text{CS}_k$  just as in the case of the ordinary Brauer centralizer algebras.

We give a few examples.

**Example 8.3.** In the following examples we always assume that  $n \geq k$  and we consider the base under the representation theoretic parametrization and we omit the subscript  $rt$ . First we calculate the product  $y_5 y_8$  when  $k = 2$ .

FIGURE 6. The product  $y_5 y_8$

Here  $y_8 = \text{inj}_{\{1,2\}} C_{\{1,2\}}$  and  $y_5 = \text{inj}_{\{1\}} \binom{2}{2} \text{pr}_{\{1\}}$ . From the formula (8.1.2), we have  $\text{pr}_{\{1\}} \text{inj}_{\{1,2\}} = (X-1)_1 \text{inj}_{\{2\}}$  (here we put  $2n+1 = X$ .) and the resulting homomorphism is  $\text{inj}_{\{1\}} \binom{2}{2} (X-1) \text{inj}_{\{2\}} C_{\{1,2\}} = (X-1) \text{inj}_{\{1\}} \text{inj}_{\{2\}} C_{\{1,2\}}$ . From the formula (8.1.3), we have  $\text{inj}_{\{1\}} \text{inj}_{\{2\}} = \text{inj}_{\{1,2\}} + \text{id}_{V_{\{1,2\}}}$  and the final result is given by the Figure 6.

Let us give a more complicated example of calculation of the product.

## 9. DUAL PAIR AND THE SPIN REPRESENTATIONS

In this section we define the subspace of the space  $\Delta \otimes \otimes^k V$ , on which the symmetric group  $\mathfrak{S}_k$  and  $\text{Spin}(2n+1)$  act as a dual pair.

From now on we always assume that  $n \geq k$  and we consider only the base under the representation-theoretic parametrization and we omit the subscript  $rt$ .

By  $I_s$ , we denote the linear subspace of  $\text{CS}_k$  spanned by the generalized Brauer diagrams, in which the number of the vertical edges (i.e.,

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$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = 3(X-2)(X-3) \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} + (X-2)(X-3)(X-4) \sum_{i=1}^5 \sum_{j=1}^5 (-1)^{i+j} z_i y_j$$

Here  $y_j$  denotes the upper row and  $z_i$  denotes the lower row given as follows.

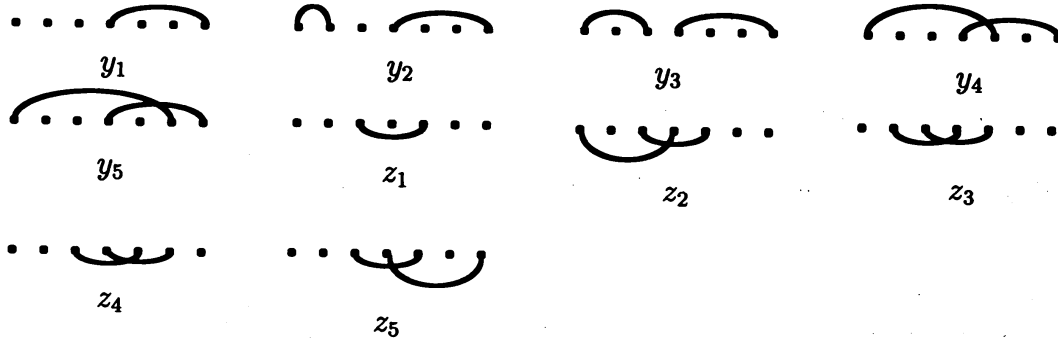


FIGURE 7. The result of the product of a more complicated example

the edges which connect the upper vertices to the lower ones) are less than and equal to  $s$ . Then  $I_s$  becomes a two sided ideal of  $\text{CS}_k$ . Then we have

$$\text{CS}_k = \mathbb{R}[\mathfrak{S}_k] \oplus I_{k-1}.$$

We define the subspace  $T_k^0$  of the space  $\Delta \otimes \otimes^k V$  by the intersection of all the kernels of the contractions  $C_{\{i,j\}}$  ( $1 \leq i < j \leq k$ ) and of the projections  $\text{pr}_{\{i_1, i_2, \dots, i_r\}}$  ( $r > 0$  and  $1 \leq i_1 < i_2 < \dots < i_r \leq k$ ).

Then two sided ideal  $I_{k-1}$  acts on this space  $T_k^0$  by 0, therefore on the space  $T_k^0$ , the symmetric group  $\mathfrak{S}_k$  and  $\text{Spin}(2n+1)$  act as a dual pair. Namely we have the following theorem.

**Theorem 9.1.** *If  $n \geq k$ , then we have*

$$(9.1.1) \quad T_k^0 = \sum_{\lambda: \text{partitions of size } k} \lambda_{\mathfrak{S}_k} \otimes [\Delta, \lambda]_{\text{Spin}(2n+1)}.$$

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